

Pf: Since $b \in (0, 1)$, we can write it in the form

$$b = \frac{1}{1+a} \quad \text{for some } a > 0$$

($\because b < 1$)

Raise to n^{th} power, apply Bernoulli's ineq.

$$b^n = \left(\frac{1}{1+a}\right)^n = \frac{1}{(1+a)^n} \leq \frac{1}{1+na}$$

By the example on Tuesday.

Let $\varepsilon > 0$ be fixed but arbitrary.

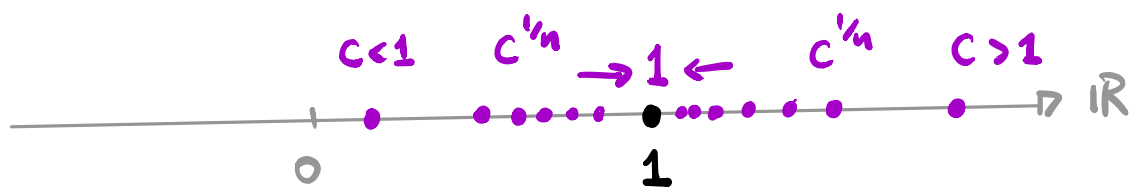
Choose $k \in \mathbb{N}$ s.t. $k > \frac{1}{a\varepsilon}$. Then $\forall n \geq k$

$$|b^n - 0| \leq \frac{1}{1+na} \leq \frac{1}{1+ka} < \frac{1}{ka} < \varepsilon$$

Example 2: Let $c > 0$ be fixed. Then,

$$\lim (c^{\frac{1}{n}}) = 1$$

Picture:



Proof: Case 1: $c = 1$ then $(c^{\frac{1}{n}}) = (1)$ const seq. ✓

Case 2: $c > 1$ ✓

Recall: $c^{\frac{1}{n}} > 1 \quad \forall n \in \mathbb{N}$. Then, for each $n \in \mathbb{N}$,

$$c^{\frac{1}{n}} = 1 + d_n \quad \text{for some } \underline{d_n > 0}$$

Raise to n^{th} power, apply Bernoulli.

$$c = (c^{\frac{1}{n}})^n = (1 + d_n)^n \stackrel{\because d_n > 0}{\geq} 1 + n d_n$$

Rearrange. $d_n \leq \frac{c-1}{n}$ _____ (*)

Let $\varepsilon > 0$ be fixed but arbitrary.

Choose $K \in \mathbb{N}$ st. $K > \frac{c-1}{\varepsilon}$.

When $n \geq K$, we have

$$|c^{\frac{1}{n}} - 1| = |d_n| = d_n \leq \frac{c-1}{n} \leq \frac{c-1}{K} < \varepsilon$$

Case 3: $0 < c < 1$

Recall: $0 < c^{\frac{1}{n}} < 1 \quad \forall n \in \mathbb{N}$.

Write: $c^{\frac{1}{n}} = \frac{1}{1 + h_n}$ where $\underline{h_n > 0}$

Raise to n^{th} power.

$$c = (c^{\frac{1}{n}})^n = \frac{1}{(1+h_n)^n} \leq \frac{1}{1+nh_n} < \frac{1}{nh_n}$$

Rearrange. $h_n < \frac{1}{cn}$ for all $n \in \mathbb{N}$

Let $\varepsilon > 0$ be fixed but arbitrary.

Choose $K \in \mathbb{N}$ st. $K > \frac{1}{c\varepsilon}$.

For $n \geq K$,

$$|c^{\frac{1}{n}} - 1| = \left| \frac{1}{1+h_n} - 1 \right| = \left| \frac{-h_n}{1+h_n} \right|$$

$$= \frac{h_n}{1+h_n} < h_n < \frac{1}{cn} \leq \frac{1}{cK} < \varepsilon$$

_____ \square

Example 3:

$$\lim (n^{\frac{1}{n}}) = 1$$

Pf: Recall: $1 \leq n^{\frac{1}{n}} \forall n \in \mathbb{N}$. Write $n^{\frac{1}{n}} = 1 + \underbrace{k_n}_{\geq 0}$

$$\Rightarrow n = (1+k_n)^n \geq 1 + \frac{1}{2}n(n-1)k_n^2$$

($\because k_n \geq 0$)

Rearrange $k_n^2 \leq \frac{n-1}{\frac{1}{2}n(n-1)} = \frac{2}{n}$.

Let $\varepsilon > 0$ be fixed but arbitrary.

Choose $K \in \mathbb{N}$ s.t. $K > \frac{2}{\varepsilon^2}$. Then $\forall n \geq K$,

$$\|n^{\frac{1}{n}} - 1\| = |k_n| = k_n \leq \sqrt{\frac{2}{n}} \leq \sqrt{\frac{2}{K}} < \varepsilon$$

_____ \square